A Multi Species Asymmetric Exclusion Process, Steady State and Correlation Functions on a Periodic Lattice

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Abstract

By generalizing the algebra of operators of the Asymmetric Simple Exclusion Process (ASEP), a multi-species ASEP in which particles can overtake each other,is defined on both open and closed one dimensional chains. On the ring the steady state and the correlation functions are obtained exactly. The relation to particle hopping models of traffic and the possibility of shock waves in open systems is discussed. The effect of the boundary condition on the steady state properties of the bulk is studied.

key words: matrix product ansatz, operator algebra, asymmetric exclusion process, traffic flow

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1 Introduction

The aim of this letter is to introduce a natural multi-species generalization of the one-species asymmetric simple exclusion process (1-ASEP) [1-5]. We will show that using the Matrix Product Ansatz (MPA) for one dimensional stochastic systems, one can algebraicly define a natural p-ASEP (p = number of species) such that all the results of 1-ASEP will be reproduced in the special case $p = 1$. We will also show that this model incorporates new features which compared with 1-ASEP makes it a better candidate as a model of traffic flow. First we resume the basic facts about 1-ASEP. In this process which is defined on a one dimensional lattice with either periodic or open boundary conditions , particles hop stochastically to their right nearest neighbor site if this site is empty and stop if this site is occupied.This model is related to many interesting physical problems including interfacial growth [6], directed polymers in random media [7,8],kinetics of biopolymerization [9], and traffic flow [10-12] or any queing problem. It is an example of a system far from equilibrium for which many exact results are known (for recent reviews see [5,13]).Some of the basic results are as follows.

On a ring of N sites where there are M particles interacting with each other, the steady state is such that all configurations have equal weights [14]. All the equal time correlation functions are also easy to calculate [14], that is

$$< \tau_i >= \frac{M}{N} \qquad < \tau_i \tau_j >= \frac{M(M-1)}{N(N-1)} \qquad \text{etc.}$$

Here the stochastic variable τ_k is 0 if site k is empty and 1 otherwise. Therefore $< \tau_k >$ is the average density and $< \tau_k \tau_l >$ is the density-density correlation. The unequal time correlation functions are not known in general and only certain quantities, like the diffusion constants have been calculated exactly [15,16].

On an open chain where particles arrive at the left with rate α and leave the right with rate β the steady state and the phase diagram [17-19] is known exactly. Particularly when $\alpha + \beta = 1$, the steady state is defined by a Bernouli measure. The average density of particles is given by α and the steady current is $J = \alpha(1 - \alpha)$. The measure being

multiplicative, there is no correlation in this case.

When $\alpha+\beta\neq 1$, the steady state is again multiplicative [19], but this time matrices replace numbers. This is the content of the matrix product ansatz (MPA)[20]. In this case the probability of a configuration $(\tau_1,\tau_2,...,\tau_N)$ is given by

$$P(\tau_1,\tau_2,...,\tau_N)=\frac{1}{Z_N}<W|D(\tau_1)D(\tau_2)...D(\tau_N)|V> \tag{1}$$

where the matrices $D:=D(1)$ and $E:=D(0)$ and the vectors $<W|$ and $|V>$ satisfy the following relations:

$$DE=D+E \qquad D|V>=\frac{1}{\beta}|V> \qquad <W|E=\frac{1}{\alpha}<W| \tag{2}$$

and Z_N is a normalization constant. (For the dynamical matrix ansatz see [21].) It can be shown that the representations of the algebra (2) are either one dimensional or infinite dimensional. The one dimensional representations are the ones which give the steady state on the ring(where the matrix element (1) is replaced by a trace),and on the open chain when $\alpha+\beta=1$. The infinite dimensional representations are used for the open chain when $\alpha+\beta\neq 1$. Although the details of the one dimensional representations are important for the properties of the steady state in the open chain, for the periodic chain where the number of particles are fixed, once the existence of a one dimensional representation is shown, it follows that all configurations are of equal weights and calculation of all equal time quantities can be done simply by using combinatorics. In [22] it has been shown that the MPA is really not an ansatz and in fact the steady state of any one-dimensional stochastic process governed by a Hamiltonian with only nearest neighbor interaction can be put in the form (1). In the general case where each variable τ_k takes $p+1$ values say $0,1,...p$, one writes again the steady state, in the same form as in (1). The p+1 operators $D_i:=D(\tau_i)$ and the vectors $|V>$ and $<W|$ should satisfy a set of algebraic relations [22] written compactly as follows:

$$h^B(\mathcal{D}\otimes\mathcal{D})=X\otimes\mathcal{D}-\mathcal{D}\otimes X \tag{3}$$

$$(h^N\mathcal{D}-X)|V>=0 \qquad\qquad <W|(h^1\mathcal{D}+X)=0 \tag{4}$$

where h^B is the part of the bulk Hamiltonian acting on two neighboring sites, h^1 and h^N are boundary terms, and $(\mathcal{D})_i = D_i$ and X are two column matrices. Usually X is a suitably chosen numerical matrix. However, finding the representations of the obtained algebra is by no means an easy problem and in fact may well be harder than the original problem. In this sence the work of Kreb and Sandow [22] implies that the MPA does not facilitates much the search for the steady state of one-dimensional stochastic systems or the ground state energies of generalized spin chains.However one can reverse the problem and turns the observation of [22] into a virtue. Thats, one can postulate a consistent algebra with nontrivial representations and then find via (3-4),if a physically meaningful process or spin Hamiltonian corresponds to this algebra. In this letter we will go through this procedure and define the p-species ASEP. We start from the following algebra

$$D_iE = \frac{1}{v_i}D_i + E \qquad i = 1...p \tag{5}$$

$$D_jD_i = \frac{v_iD_j - v_jD_i}{v_i - v_j} \qquad j > i \tag{6}$$

where the parameteres v_i are finite non-zero real numbers which we order as $v_1 \leq v_2 \leq .... \leq v_p$. It is easy to check that this algebra is associative, thats for any $k > j > i$, $D_j(D_iE) = (D_jD_i)E$ and $D_k(D_jD_i) = (D_kD_j)D_i$. For details of the derivation and also for the representations see [23]. It is now straightforward to check that the following Hamiltonian when inserted in (3) gives the above algebra,

$$h^B = -\sum_{i=1}^{p} v_i(E_{0i} \otimes E_{i0} - E_{ii} \otimes E_{00}) - \sum_{j>i}^{p}(v_j - v_i)(E_{ij} \otimes E_{ji} - E_{jj} \otimes E_{ii}) \tag{7}$$

where the choice made for X is as follows: $x_0 = -1, x_i = v_i/p$, and $\mathcal{D}_0 = E, (\mathcal{D})_i = \frac{D_i}{p}$.The factors $\frac{1}{p}$ are for later convenience. Here the matrices E_{ij} have the standard definition $(E_{ij})_{kl} = \delta_{i,k}\delta_{j,l}$. The local Hilbert space of each site is spanned by the vectors $|0>, |1>, ...|p>$. The state $|0>$ means that the corresponding site is vacant and the state $|j>$ means that it is occupied by a particle of type j. The Hamiltonian (7) describes a process in which particles of type j hop with rate v_j and when they encounter particles of type i, with $v_i < v_j$ they interchange their sites, as if fast particles stochastically overtake

slow particles with a rate $v_j - v_i$. This model seems to be very natural as a simple model of one-way traffic flow. Real models of traffic flow are too complex to be amenable to analytical treatment. One species ASEP is a very simple model of traffic which has been extensively studied in the past few years by analytical methods. Compared with 1-ASEP, our model,while still based on a nice algebraic structure and hence amenable to analytical treatment, incorporates new more realistic features of traffic flow, namely the existence of different intrinsic speeds for the cars and the possibility of overtaking or passing [24]. Moreover, while 1-ASEP is a suitable model for one-lane traffic flow the present model, at least in those situations where the probability of cars riding side by side is small, seems to be a good model for multi-lane traffic flow. Note that when fast particles with speed of say v_2 reach slow ones with speed $v_1 < v_2$ they pass them only stochastically. That is,in a time interval dt a fraction $(v_2 - v_1)dt$ of the fast cars overtake and the rest are stopped behind the slow ones. This means that although a fast car has an intrinsic speed $v_2 > v_1$ in an empty road,its effective speed depends on both the distance and the relative speed of the car ahead. Considering a two particle system (a fast one with coordinate x_2 behind a slow one with coordinat x_1) we can write :

$$v_2{}^{eff} := \frac{d}{dt} < x_2 >= v_2 - v_1 P(-1,t) \tag{8}$$

where $P(-1,t)$ is the probability of the fast particle being one lattice site behind the slow particle. The above equation can be obtained after lenghty calculations starting from the master equation of the two particle system in coordinate space, although it is obvious from the very definiton of the process. It is intuitively clear that $P(-1,t)$ is an increasing function of $v_2 - v_1$ and a decreasing function of the initial seperation $x_1 - x_2$. which implies a realistic behaviour for $v_2{}^{eff}$.

Moreover we have shown elsewhere [23] that the same algebra (5,6) describes p-ASEP on an open system where particles of speed v_i enter the left with rate $\frac{\alpha v_i}{p}$ and leave the right with rate $v_i + \beta - 1$. In the open system the unit of time is set so that the average speed of particles is unity. Hence α and β are the total arrival and average departure rates respectively. The forms of these rates and their dependence on speed again is a natural

property of highway traffic flow. So much for the relevance of this model to traffic flow.

2 Correlation Functions

In the following we restrict ourselves to the periodic boundary condition and calculate the steady state and some equal-time correlation functions.

The algebra (5,6) has a one parameter family of one dimensional representations, namely $E = \frac{1}{\alpha}, \quad D_i = \frac{v_i}{v_i - \alpha}$ where α is a free parameter. This means that in the steady state all configurations have equal weights. If there are m_i particles of type i, and the total number of particles is $M = m_1 + m_2 + ...m_p$,then the probability of all the configurations are equal to :

$$C = \frac{1}{N!} m_1! m_2! ... m_p! (N - M)! \tag{9}$$

We define the following functions:

$$\begin{aligned} n^{(i)}(\tau_k) &:= \delta_{i,\tau_k} \\ n(\tau_k) &:= n^{(1)}(\tau_k) + n^{(2)}(\tau_k) + ...n^{(p)}(\tau_k) \\ v(\tau_k) &:= v_1 n^{(1)}(\tau_k) + v_2 n^{(2)}(\tau_k) + ...v_p n^{(p)}(\tau_k) \end{aligned} \tag{10}$$

The one point functions $< n_k^{(i)} >, < n_k >$ and $< v_k >$ of these quantities give respectively the average number density of particles of type (i),the average number density of all types of particles and the average intrinsic speed of particles at site k. Due to the uniformity of the measure none of the correlation functions will depend on the site indices. It is simple to show the following:

$$< n_q^{(i)} >= \frac{m_i}{N}, \qquad < n_q^{(i)} n_r^{(j)} >= \begin{cases} \frac{m_i(m_i-1)}{N(N-1)} & i = j \\ \frac{m_i m_j}{N(N-1)} & i \neq j \end{cases} \tag{11}$$

Higher correlation functions have similar forms (i.e $< n_q^{(i)} n_r^{(i)} n_s^{(i)} >= \frac{m_i(m_i-1)(m_i-2)}{N(N-1)(N-2)}$). As a sample calculation we note that:

$$< n_q^{(i)} >\equiv< n_1^{(i)} >= \sum_{\{\tau_\alpha\}} \delta_{\tau_1,i} P(\tau_1, ..., \tau_N) = C \sum_{\tau_2,\tau_3,...\tau_N} 1 \tag{12}$$

The last sum is the number of ways the rest of particles (except (i)) can be distributed on the ring. Combining this with the value of C given in (9) gives the result. From (11) the following quantities are calculated:

$$< n_q >= \frac{M}{N} \qquad < v_q >= \frac{M}{N} < v > \qquad < n_q n_r >= \frac{M(M-1)}{N(N-1)} \tag{13}$$

$$< v_q v_r >= \frac{M^2 < v >^2 - M < v^2 >}{N(N-1)} \tag{14}$$

Here the averages $< v >, and < v^2 >$ are taken with respect to the population present in the system, i.e:

$$< v >:= \frac{m_1 v_1 + m_2 v_2 + ... m_p v_p}{M}$$

while the left hand side averages are taken with respect to the steady state configurations of the particles on the ring. The result (14) is the new quantity which can be defined in this process and is absent in 1-ASEP. From (13) and (14) one obtains the correlation function for intrinsic speeds (or in fact types of particles) at different sites:

$$g(\rho) :=< v_k v_l > - < v_k >< v_l >= \frac{\rho}{N-1}(\rho < v >^2 - < v^2 >) \tag{15}$$

where $\rho = \frac{M}{N}$ is the density . It is seen that the maximum correlation exists at $\rho = 0$ and $\rho = \frac{<v^2>}{<v>^2}$ and the minimum correlation at $\rho = \frac{<v^2>}{2<v>^2}$ All the other correlation functions can be calculated in this manner.For example some lenghty calculation will give

$$< v_q v_r v_s >= \frac{M^3 < v >^3 - 3M^2 < v >< v^2 > + 2M < v^3 >}{N(N-1)(N-2)} \tag{16}$$

3 Boundary Induced Negative Currents

An intersting physical quantity to consider is the average current of each type of particles. From the form of the process one can write the following continuity equation for the density of each type of particles.

$$\frac{d}{dt} < n_k^{(i)} >=< J_k^{(i)} > - < J_{k+1}^{(i)} >$$

where the current of particles of type (i) is given by:

$$< J_k^{(i)} >= v_i < n_{k-1}^{(i)}\epsilon_k > + \sum_{j<i}(v_i - v_j) < n_{k-1}^{(i)} n_k^{(j)} > - \sum_{j>i}(v_j - v_i) < n_{k-1}^{(j)} n_k^{(i)} > \quad (17)$$

and $\epsilon_k = 1 - \sum_i n_k^{(i)}$. In fact $< n_{k-1}^{(i)}\epsilon_k >$ is the probability of site k-1 being filled with an (i) particle and site k being empty. . From (14,17) the average current is calculated to be:

$$< J^{(i)} >= \frac{m_i}{N(N-1)}\Big(Nv_i - M < v > \Big) \quad (18)$$

Finally we obtain the total current as :

$$< J >:=< J^{(1)} + J^{(2)} + ...J^{(p)} >= \frac{M(N-M)}{N(N-1)} < v > \quad (19)$$

again in accord with the 1-ASEP result.

The interesting point is that for those particles whose hopping rates are less than $\frac{M}{N} < v >$, the currents $< J^{(i)} >$ turns out to be negative. One may think that this result is expectable, once we remind of the exchange of particles which takes place in the process i.e; particles ordinarily hop forward but when encountered by faster particles from the left they hop backward. However negative currents are induced only on the ring and it has been proved rigourously [23] that in an open system all the currents are proportioanl to one single current. This is an example of how in a nonequilibrium situation boundary conditions drastically affect the behaviour in the bulk. A simple way for controlling negative currents by varying the density is as follows: As far as $M < M_c := \frac{Nv_{min}}{<v>}$ where v_{min} is the lowest speed of the particles there is no negative current in the system. One can now add particles of speed $< v >$ to the system which increases only M without changing $< v >$. For all $M > M_c$ negative currents will be developed in the system.

We would like to stress that the merits of this inherent partial asymmetry are quite remarkable. On the one hand, one is tempted to think that this model may also have some relevance as a two-way traffic model and on the other hand, the algebra that describes this process is much simpler than the commutator-like algebras which one may try to define for a multi-species partialy asymmetric process from the begining. In fact

the attempted algebras of this later type [25,26] do not have even a general and elegant form for arbitrary p.

4 A note on Shock Waves

For the following discussion we consider open boundary conditions. In this case the one dimensional representations give the steady state for the case $\alpha+\beta=1$ [23]. The steady state is a translationaly invariant Bernouli measure in which the density of particles is constant along the chain. In the $p \longrightarrow \infty$ limit, where the speeds of the particles are taken from a distribution $P(v)$, the total density and the total currents are given by [23]:

$$\rho(\alpha) = \frac{\Delta(\alpha)}{\frac{1}{\alpha}+\Delta(\alpha)} \qquad J(\alpha) = \frac{1}{\frac{1}{\alpha}+\Delta(\alpha)} \tag{20}$$

where

$$\Delta(\alpha) = \int_{\alpha}^{\infty} \frac{v}{v-\alpha} P(v) dv \tag{21}$$

The parameter $\alpha > 0$ when eliminated between the above expressions gives J as a function of ρ. We will show that for a large class of distributions, $J(\rho)$ is a convex function. The only condition that $P(v)$ should satisfy is that $\lim_{v\longrightarrow\alpha} \frac{P(v)}{(v-\alpha)^2} = 0$. To prove this we rewrite (20) as:

$$J = \alpha(1-\rho) \qquad \frac{\rho}{(1-\rho)} = \alpha \int_{\alpha}^{\infty} \frac{v}{v-\alpha} P(v) dv \tag{22}$$

from which $\frac{d^2J}{d\rho^2} = \frac{d^2\alpha}{d\rho^2}(1-\rho) - 2\frac{d\alpha}{d\rho}$. Denoting the integral $\int_{\alpha}^{\infty} \frac{v}{(v-\alpha)^k} P(v) dv$ by I_k; noting that $\frac{d}{d\alpha} I_k = k I_{k+1}$ for $k = 1, 2$ and differentiating the second relation of (22) twice with respect to ρ, we find after some algebra

$$\frac{d^2J}{d\rho^2} = -2 \frac{1}{(1-\rho)^3} \frac{I_2 + \alpha I_3}{(I_1+\alpha I_2)^2} \tag{23}$$

which is always negative. This then hints and only hints at the possiblity of shock wave solutions for the equaion $\frac{\partial\rho}{\partial t} + \frac{\partial J}{\partial x} = \frac{\partial\rho}{\partial t} + J'(\rho)\frac{\partial\rho}{\partial x} = 0$, the simplest of which are in the form of step functions with density ρ_- at the left and ρ_+ at the right of the front which itself moves with the speed $V_{shock} = \frac{J_- - J_+}{\rho_- - \rho_+}$. In conclusion, the approach presented

in this letter, that is, begining from consistent algebras and then finding the process may be further pursued to study other one dimensional reaction diffusion processes. It may also be possible to map this model to more complex interface growth models. In this letter we have been lucky to postulate an algebra which while being simple and beautiful,approximately models traffic, an every-day life phenomenon.

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